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Acceleration of Convergence of One Iterative Method for Finding the Roots of Equations

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Abstract. In this paper we consider two iterative methods which accelerate the finding of the roots of equations.

It is well known that, if the equation

$$x = f(x) \tag{1}$$

has only one root $x = r$ in the interval $[a, b]$, and if the derivative $f'(x)$ of the function $f(x)$ satisfies the condition

$$\max \left|f'(x)\right| = M < 1, \quad \text{for } x \in [a, b], \tag{2}$$

then the iterative method

$$x_{k+1} = f(x_k), \quad k = 0, 1, 2 \ldots, \tag{3}$$

converges to the root $x = r$ of the equation (1), where the initial value x_0 can be any number from the interval $[a, b]$. The convergence of method (3) is more rapid if M has a small value.

In this paper we consider the values $f'(a)$ and $f'(b)$ and use it to determine $\max \left|f'(x)\right|$.

We consider two iterative methods for finding the root $x = r$ of the equation (1) in cases when

$$f'(a) \neq 0, \quad f'(b) \neq 0 \tag{4}$$

$$\max \left|f'(x)\right| = M < 1 \tag{5}$$

and when $f'(x)$ is increasing or decreasing.

If the conditions (4) and (5) are satisfied, then

$$\left|f'(a)\right| < 1 \quad \text{and} \quad \left|f'(b)\right| < 1. \tag{6}$$

We put

$$f'(a) = \alpha, \quad f'(b) = \beta. \tag{7}$$

Because of (6) and (7), we have

$$1 - \alpha > 0 \quad \text{and} \quad 1 - \beta > 0. \tag{8}$$

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If the function $f'(x)$ is increasing on the interval $[a,b]$, then we consider the following two cases

$$(C_1) \qquad -1 < \alpha \le f'(x) \le \beta < 0,$$

$$(C_2) \qquad 0 < \alpha \le f'(x) \le \beta < 1.$$

In these cases, we write the equation (1) in the form

$$x = \frac{1}{1-\alpha}\big(f(x) - \alpha x\big), \tag{9}$$

that is, in the form

$$x = f_1(x), \tag{10}$$

where

$$f_1(x) = \frac{1}{1-\alpha}\big(f(x) - \alpha x\big). \tag{11}$$

From (11) we obtain

$$f_1'(x) = \frac{1}{1-\alpha}\big(f'(x) - \alpha\big). \tag{12}$$

As $f'(x)$ is an increasing function, considering (8), we conclude from (12) that the function $f_1'(x)$ is also increasing.

For the case (C_1), we have

$$\max\big|f'(x)\big| = |\alpha|. \tag{13}$$

Considering (7), we obtain from (12)

$$f_1'(a) = 0, \quad f_1'(b) = \frac{\beta-\alpha}{1-\alpha} < \frac{|\alpha|}{1+|\alpha|} < |\alpha|. \tag{14}$$

From (14) we conclude that

$$\max\big|f_1'(x)\big| = \frac{\beta-\alpha}{1-\alpha} < \frac{|\alpha|}{1+|\alpha|} < |\alpha|. \tag{15}$$

For the case (C_2), we have

$$\max\big|f'(x)\big| = \beta. \tag{16}$$

Considering (7), we obtain from (12)

$$f'(a) = 0, \quad f'(b) = \frac{\beta-\alpha}{1-\alpha}. \tag{17}$$

From (17) we conclude that

$$\max\big|f_1'(x)\big| = \frac{\beta-\alpha}{1-\alpha} < \beta. \tag{18}$$

When $f'(x)$ is decreasing, we consider the other two cases

$$(C_3) \qquad 0 > \alpha \ge f'(x) \ge \beta > -1,$$

$$(C_4) \qquad 1 > \alpha \ge f'(x) \ge \beta > 0.$$

In these cases, we write the equation (1) in the form

$$x = \frac{1}{1-\beta}\big(f(x) - \beta x\big), \tag{19}$$

that is, in the form

$$x = f_2(x), \tag{20}$$

where

$$f_2(x) = \frac{1}{1-\beta}\big(f(x) - \beta x\big). \tag{21}$$

From (21) we obtain

$$f_2'(x) = \frac{1}{1-\beta}\big(f'(x) - \beta\big). \tag{22}$$

As $f'(x)$ is a decreasing function, considering (8), we conclude from (22) that the function $f_2'(x)$ is also decreasing.

For the case (C_3), we have

$$\max\big|f'(x)\big| = |\beta|. \tag{23}$$

In view of (7), we obtain from (22)

$$f_2'(a) = \frac{\alpha-\beta}{1-\beta} < \frac{|\beta|}{1+|\beta|} < |\beta|, \quad f_2'(b) = 0. \tag{24}$$

From (24) we conclude that

$$\max\big|f_2'(x)\big| = \frac{\alpha-\beta}{1-\beta} < |\beta|. \tag{25}$$

For the case (C_4), we have

$$\max\big|f'(x)\big| = \alpha. \tag{26}$$

In view of (7), we obtain from (22)

$$f_2'(a) = \frac{\alpha-\beta}{1-\beta} < \alpha, \quad f_2'(b) = 0. \tag{27}$$

From (27) we conclude that

$$\max\big|f_2'(x)\big| = \frac{\alpha-\beta}{1-\beta} < \alpha. \tag{28}$$

From the conditions (4), (5) and (7), as well as from (13), (15) and (16), (18), we see that

$$\max\big|f_1'(x)\big| < \max\big|f'(x)\big| \tag{29}$$

when $f'(x)$ is increasing, which satisfies (C_1) or (C_2). Then, from (9) we obtain the iterative method

$$x_{k+1} = \frac{1}{1-\alpha}\big(f(x_k) - \alpha x_k\big), \quad k = 0, 1, 2 \ldots . \tag{30}$$

Analogously, from the conditions (4), (5) and (7), as well as from (23), (25) and (26), (28), we see that

$$\max\left|f_2'(x)\right| < \max\left|f'(x)\right| \tag{31}$$

when $f'(x)$ is decreasing, which satisfies (C_3) or (C_4). Then from (19) we obtain the iterative method

$$x_{k+1} = \frac{1}{1-\beta}\big(f(x_k) - \beta x_k\big), \quad k = 0, 1, 2 \ldots . \tag{32}$$

Because of (29) and (31), the method (30) or the method (32) converges more rapidly than the method (3).

We demonstrate the methods (31) and (32) on the following examples.

Example 1. *The equation*

$$x^3 - 8x + 5 = 0 \tag{A}$$

has only one root $x = r$ in the interval $[2, 3]$.

We can write the equation (A) in the form

$$x = \frac{8}{x} - \frac{5}{x^2}, \tag{A_1}$$

that is, in the form

$$x = f(x), \tag{A_2}$$

where

$$f(x) = \frac{8}{x} - \frac{5}{x^2}. \tag{A_3}$$

From (A_3) we obtain

$$f'(x) = -\frac{8}{x^2} + \frac{10}{x^3}. \tag{A_4}$$

On $[2, 3]$ *the function $f'(x)$ is increasing, and*

$$f'(a) = f'(2) = \alpha = -\frac{3}{4} = -0.75, \quad f'(b) = f'(3) = \beta = -\frac{14}{27}, \tag{A_5}$$

which means that we can apply the formula (30) in the case (C_1). According to (13), from (A_5) we have

$$\max\left|f'(x)\right| = |\alpha| = 0.75$$

and according to (15), we have from (A_5)

$$\max\left|f_1'(x)\right| = \frac{\beta-\alpha}{1-\alpha} = 0.132275132.$$

Now, the formul;a (30) is reduced to

$$x_{k+1} = \frac{1}{1.75}\left(\frac{8}{x_k} - \frac{5}{x_k^2} + 0.75x_k\right), \quad k = 0, 1, 2, \ldots, \tag{A_6}$$

and the formula (3) to

$$x_{k+1} = \frac{8}{x_k} - \frac{5}{x_k^2}, \quad k = 0, 1, 2 \dots . \tag{A_7}$$

The values x_k are listed in Table 1. Both formulas (A_6) and (A_7) start from the same initial value $x_0 = 3$.

TABLE 1.

Formula (A_6)	*Formula* (A_7)
$x_0 = 3$	$x_0 = 3$
$x_1 = 2.492063492$	$x_1 = 2.111111111$
	$x_2 = 2.667590028$
$x_2 = 2.442362884$	$x_3 = 2.296323254$
	....
$x_3 = 2.439477086$	$x_5 = 2.377364738$
	....
$x_4 = 2.439320604$	$x_{10} = 2.446868841$
	....
$x_5 = 2.439312154$	$x_{20} = 2.439422287$
	....
$x_6 = 2.439311698$	$x_{30} = 2.439313292$
	....
$x_7 = 2.439311673$	$x_{40} = 2.439311695$
	....
	$x_{49} = 2.439311671$
$x_8 = 2.439311672$	$x_{50} = 2.439311672$
$x_9 = 2.439311672$	$x_{51} = 2.439311672$

Example 2. *The equation*

$$x + e^x - 2 = 0 \tag{B}$$

has only one root $x = r$ in the interval $[0, 0.8]$.

We can write the equation (B) in the form

$$x = \ln(2 - x), \tag{B_1}$$

that is, in the form

$$x = f(x), \tag{B_2}$$

where

$$f(x) = \ln(2 - x). \tag{B_3}$$

From (B_3) we obtain

$$f'(x) = \frac{1}{x - 2}. \tag{B_4}$$

On $[0, 0.8]$ *the function $f'(x)$ is decreasing, and*

$$f'(a) = f'(0) = \alpha = -\frac{1}{2}, \quad f'(b) = f'(0.8) = \beta = -\frac{5}{6}, \tag{B_5}$$

which means that we can apply the formula (32) in the case (C_3). *According to (23), from* (B_5) *we have*

$$\max\left|f'(x)\right| = |\beta| = \frac{5}{6}$$

and according to (25), we have from (B_5)

$$\max\left|f_2'(x)\right| = \frac{\beta-\alpha}{1-\beta} = 0.181818182.$$

Now, the formula (32) is reduced to

$$(B_6) \qquad x_{k+1} = \frac{1}{11}\big(6\ln(2-x_k)+5x_k\big), \quad k=0,1,2,\dots,$$

and the formula (3) to

$$(B_7) \qquad x_{k+1} = \ln(2-x_k), \quad k=0,1,2\dots.$$

The values x_k *are listed in Table 2. Both formulas* (B_6) *and* (B_7) *start from the same initial value* $x_0 = 0.8$.

TABLE 2.

Formula (B_6)	*Formula* (B_7)
$x_0 = 0.8$	$x_0 = 0.8$
$x_1 = 0.463084485$	$x_1 = 0.182321557$
$x_2 = 0.444917036$	$x_2 = 0.597560106$
$x_3 = 0.44306896$	$x_3 = 0.338213501$
$x_4 = 0.442876765$	$x_5 = 0.400189062$
$x_5 = 0.442856732$	$x_{10} = 0.447472609$
$x_6 = 0.442854644$	$x_{20} = 0.442909554$
$x_7 = 0.442854426$	$x_{30} = 0.442853978$
$x_8 = 0.442854404$	$x_{40} = 0.442854409$
$x_9 = 0.442854401$	$x_{46} = 0.442854402$
$x_{10} = 0.442854401$	$x_{47} = 0.442854401$
	$x_{48} = 0.442854401$

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